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Example 1 Ions

familiar: $P_I = n_I \cdot k_B \cdot T$

#density

Boltzmann constant

Maxwell-Boltzmann

$$n(p)\,dp = \frac{n_I \cdot 4\pi p^2 dp}{(2\pi m_I k_B T)^{3/2}} e^{-p^2/2m_I kT}$$

What is n_I?

$$n_I = \sum_{i\,(\text{species})} n_i = \sum_i \frac{\rho}{m_u} \cdot \frac{X_i}{A_i}$$

mass fraction

mass of ion / m_u

often we define: $Y_i = \frac{X_i}{A_i}$ "mol/g"

Def: mean molecular weight of Ions:

$$\frac{1}{\mu_I} = \sum_i \frac{X_i}{A_i} \longrightarrow n_I = \frac{\rho}{\mu_I \cdot m_u}$$

Gas Constant $R = k_B/m_u$

$$P_I = \frac{R}{\mu_I} \cdot \rho \cdot T$$

EXAMPLES –

Electrons

$$P_e = n_e k_B T$$

$$n_e = \sum_i z_i n_i = \frac{\rho}{m_u} \sum_i X_i \frac{z_i}{A_i} = \frac{\rho}{m_u} \sum z_i Y_i$$

Def: Mean molecular weight per electron

$$\frac{1}{\mu_e} \equiv \sum_i z_i \frac{X_i}{A_i} = \sum_i z_i Y_i$$

$$\rightarrow n_e = \rho / \mu_e \cdot m_u$$

Example

Assume we have: ^1H: $A=1, z=1$

Rest: $A \approx 2 \cdot z$

$\rightarrow$ derive formula for μ_e (or $\frac{1}{\mu_e}$)

Result: $\frac{1}{\mu_e} = X + (1-X) \cdot \frac{1}{2} = \frac{1}{2}(1+X)$

$$P_e = \frac{\mathcal{R}}{\mu_e} \cdot \rho \cdot T$$

Total [Gas]

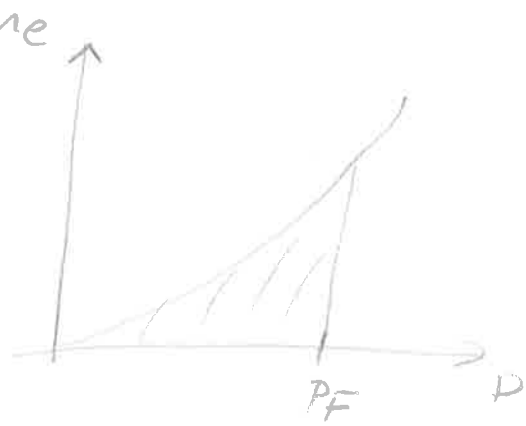
$$P_{Gas} = P_I + P_e = \mathcal{R} T \rho \underbrace{\left(\frac{1}{\mu_e} + \frac{1}{\mu_I}\right)}_{= \frac{1}{\mu}} = \frac{\mathcal{R} T \rho}{\mu}$$

$$\boxed{\frac{1}{\mu} = \frac{1}{\mu_e} + \frac{1}{\mu_I}}$$

Degenerate e^-

Heisenberg: $\Delta V \cdot \Delta^3 p \gtrsim h^3$

$$n_e(p)\,dp = \frac{2}{h^3} \cdot 4\pi p^2\,dp$$



Fermi momentum

$$P_F:\ n_e = \int_0^{P_F} n_e(p)\,dp = \frac{8\pi}{3h^3} P_F^3 \overset{!}{=} \frac{\rho}{\mu_e \cdot m_u}$$

$$\longrightarrow\ P_F = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3}$$

using $v = P/m_e$ | $P_{e,deg} = \frac{1}{3}\int_0^{P_F} n_e(p) \cdot \frac{P}{m_e} \cdot P\,dp = \ldots$

$$P_{e,deg} = \frac{8\pi}{15 m_e h^3} P_F^5 = \underbrace{\frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3} \frac{1}{m_u^{5/3}} \left(\frac{\rho}{\mu_e}\right)^{5/3}}_{=:\ K_1 \cdot \rho^{5/3}}$$

rel. deg: $v \longrightarrow c$

$(E \gg m_e c^2,\ v = c \cdot \sqrt{\frac{p^2}{m^2 c^2 + p^2}}$

as $p \gg m_e c$)

$$P_{e,rel\text{-}deg} = \frac{1}{3}\int_0^{P_F} n_e(p) \cdot c \cdot p = \underbrace{\frac{h \cdot c}{8}\left(\frac{3}{\pi}\right)^{1/3} \frac{1}{m_u^{4/3}} \left(\frac{\rho}{\mu_e}\right)^{1/3}}_{=:\ K_2 \cdot \rho^{4/3}}$$

Q: Can we compute u?

Q: When do we make transition from rel to non-rel? at what density?