RECAP / SUMMARY 20160314-1

STELLAR STRUCTURE EQUATIONS

t-indep t-dep

1) $$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}$$

2) $$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} - \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2}$$

3) $$\frac{\partial l}{\partial m} = \varepsilon_{nuc} + \varepsilon_\nu - c_P \frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial P}{\partial t}$$

4) $$\frac{\partial T}{\partial m} = \frac{GmT}{4\pi r^4 P}\nabla \times \left[1 + \frac{r^2 \partial^2 r}{Gm\,\partial t^2}\right]$$

5) $$\frac{\partial X_i}{\partial t} = f_i(\rho, T, \vec{X})$$

where $\vec{X} = (x_1, x_2, x_3, \ldots)$ chemical species

Q: if we add 2 transport processes: will gradient become steeper or shallower?

$$\nabla = \begin{cases} \nabla_{rad} = \left.\frac{d\ln T}{d\ln P}\right)_{RAD} = \frac{3}{16\pi a c G}\frac{\kappa l P}{m T^4} & \text{rad. regions} \\ \nabla_{ad} = \left.\frac{\partial \ln T}{\partial \ln P}\right)_{ad} & \text{good Approximation in efficient convection regions, semi conv.} \end{cases}$$

USUALLY:

$$\min\{\nabla_{ad}, \nabla_{rad}\} \le \nabla \le \max\{\nabla_{ad}, \nabla_{rad}\}$$

Lane Emden EQUATION

Simple Assumptions

- T, ρ, P decrease outward
- chemical homogeneity
- l increases outward
- spherical symmetry

→ try stationary, time-independent solution!

Boundary conditions: (needed to solve PDE system)

for r, m, l

We use – in centre: $r=0$, $m=0$, $l=0$

– at surface: $r=R$, $m=M$, $l=L$

at surface: additional

effective temperature:

DEL $L = 4\pi\sigma R_{eff}^2 T_{eff}^4$

EOS (in general, typical)

$$P = \frac{R}{\mu_I} \rho T + P_e + \frac{1}{3} a T^4$$

P_e: electron pressure - electron gas:
- ideal gas
- non-rel deg. gas
- rel-deg gas
- rel.-non-deg gas [high T] $\rightarrow e^+e^-$ pairs

Simplifications: EQN 1, 2, 4 only couple implicitly through dependence on $P(T, \rho)$

$\rightarrow$ Assume P depends on ρ (T-dep implicit!)

Assume Power law $P = K \cdot \rho^\gamma$ (recall e^- deg press.)

NOTE: This "γ" is not an adiabatic index!
because we can have changes in S
(and generally will have)
γ here is called the POLYTROPIC INDEX of the model

DEF POLYTROPIC INDEX $\gamma = 1 + \frac{1}{n} = \frac{n+1}{n}$

Polytropic Stellar Models (hydrostatic)

- What is a polytrope? $P = K \cdot \rho^{\gamma}$

specific to model $P(r) = K \cdot \rho(r)^{\gamma}$

$\rightarrow$ T-dep implied, with constant K, γ

not independent ← $\boxed{\text{Q: How?}}$

Note: Different from Polytropic EOS: $P = K \cdot \rho^{\gamma}$

Start: $\dfrac{dP}{dr} = -\rho \dfrac{Gm}{r^2} \quad \Big| \times \dfrac{r^2}{\rho}$

$$\frac{r^2}{\rho} \frac{dP}{dr} = -Gm \quad \Big| \frac{d}{dr}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho} \frac{dP}{dr}\right) = -G \frac{dm}{dr} \quad \Big| \frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{1}{r^2} \frac{d}{dr}\left(\frac{r^2}{\rho} \frac{dP}{dr}\right) = -4\pi G \rho$$

use Polytrope $P = K \cdot \rho^{\gamma}$, $\gamma = 1 + \frac{1}{n}$

Q: what are typical values of n?

Example: Fully convective star:

- of ideal gas: $3/2$
- of rel. deg. gas 3

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$$\frac{(n+1)K}{4\pi G n} \cdot \frac{1}{r^2}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) = -\rho$$

the function $\rho(r)$ is called the POLYTROPE

$0 \leq r \leq R$

Boundary conditions:

- surface: $R = r \longrightarrow \rho = 0$ because $P(R) = 0$
- centre: $\frac{dP}{dr} = 0 \longrightarrow \left.\frac{d\rho}{dr} = 0 \right|_{r=0}$

$\rightarrow$ uniquely defined by parameters K, n, R

ANSATZ: DEF: Θ ("theta") $\ni 0 \leq \Theta \leq 1$

with $\rho = \rho_c \cdot \Theta^n$

$$\rightarrow \underbrace{\left[\frac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}}\right]}_{=: \alpha^2} \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Theta}{dr}\right) = -\Theta^n$$

DEF: $r = \alpha \cdot \xi$

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\Theta}{d\xi}\right) = -\Theta^n}$$

LANE-EMDEN EQUATION

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Boundary Conditions in new variables

Surface: $\Theta \stackrel{!}{=} 0$

Centre: $\frac{d\Theta}{d\xi} = 0 \quad (\xi = 0), \quad \Theta = 1$

Value of ξ @ surface? need to find!

ξ_1 defined such that $\Theta(\xi_1) = 0$

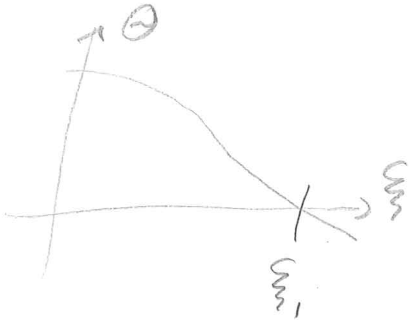
Radius of Star

$$M = \int_0^R 4\pi r^2 \rho \, dr = 4\pi \alpha^3 \rho_c \int_0^{\xi_1} \xi^2 \Theta^n \, d\xi$$

USE L.E.Q:

$$M = -4\pi \alpha^3 \rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\Theta}{d\xi}\right) d\xi$$

$$\boxed{M = -4\pi \alpha^3 \rho_c \, \xi_1^2 \left.\frac{d\Theta}{d\xi}\right|_{\xi_1}}$$



Polytropic Constants:

SOLUTION CONST. OF LE FOR GIVEN n

$$R_n = \xi_1, \quad M_n = -\xi_1^2 \left(\left.\frac{d\Theta}{d\xi}\right|_{\xi_1} \right) > 0$$

$$\rightarrow R = R_n \cdot \alpha, \quad M = \rho_c 4\pi\alpha^3 M_n$$