Recap / Summary 20170313-1-

Stellar Structure Equations

t-indep t-dep

1) $\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}$

2) $\frac{\partial P}{\partial m} = \frac{Gm}{4\pi r^4} \qquad - \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2}$

3) $\frac{\partial l}{\partial m} = \varepsilon_{nuc} + \varepsilon_\nu \qquad - c_p \frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial P}{\partial t}$

4) $\frac{\partial T}{\partial m} = \frac{GmT}{4\pi r^4 P}\nabla \times \left[1 + \frac{r^2}{Gm}\frac{\partial^2 r}{\partial t^2}\right]$

5) $\frac{\partial X_i}{\partial t} = f_i(\rho, T, \vec{X})$

where $\vec{X} = (X_1, X_2, X_3, \ldots)$ chemical species

$\nabla = \nabla_{rad} = \left.\frac{d\ln T}{d\ln P}\right|_{RAD} = \frac{3}{16\pi a c G}\frac{\kappa l P}{m T^4}$ rad regions

$\nabla_{ad} = \left.\frac{\partial \ln T}{\partial \ln P}\right|_{AD}$ good approximation for efficient convective regions, semi convection

Q: If we add 2 transport processes will gradient become steeper or shallower?

Q: why ∇_{ad} in SC but not TH?

USUALLY: $\min\{\nabla_{ad}, \nabla_{rad}\} \leq \nabla \leq \max\{\nabla_{ad}, \nabla_{rad}\}$

Lane Emden Equation

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Simple assumptions:

- T, ρ, P decrease outward
- chemical homogeneity
- l increases outward
- sperical symmetry

→ try stationary, time-independent solution

Boundary Conditions (need to solve PDE system)
for r, m, l

We use: – in centre: $r=0$, $m=0$, $l=0$
– at surface: $r=R$, $m=M$, $l=L$
(L, R may be results rather than input)

– at surface, additionally:
- effective temperature (connecting L, R, T)

DEF: $L = 4\pi\sigma R_{eff}^2 T_{eff}^4$

EOS (typical, moderately general case)

$$P = \frac{R}{\mu_I} \rho T + P_e + \frac{1}{3} a T^4$$

P_e: electron pressure - electron gas
- ideal gas
- non-rel. deg gas
- rel.-deg gas
- rel non-deg gas [high T] $\rightarrow e^+e^-$ pairs

[Q: Where & why may this decomposition fail?

Simplifications: EQN 1, 2, 4 only couple implicitly through dependence on $P(T, \rho)$ - EOS

$\rightarrow$ Assume P depends on ρ only
(T-dep. implicit!)

assume power law $P = K \cdot \rho^{\gamma}$
(recall this form from deg. e^- pressure!)

NOTE: The "γ" in the Polytrope is not an adiabatic index!!! - because we can have changes in S - and in general will

γ is called the polytropic exponent

DEF $\gamma = 1 + \frac{1}{n} = \frac{n+1}{n}$ where n is the Polytropic INDEX of the model

Polytropic Stellar Models

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(hydrostatic)

- What is a Polytrope?

$P = K \cdot \rho^{\gamma}$

- specific to model

$P(r) = K \cdot \rho(r)^{\gamma}$ ← const

→ T-dep implied not independent [Q: how?]

with constant K, γ

NOTE Different from Polytropic EOS: $P = K \cdot \rho^{\gamma}$

START: $$\frac{dP}{dr} = -\rho \frac{Gm}{r^2} \quad \Big| \times \frac{r^2}{\rho}$$

$$\frac{r^2}{\rho}\frac{dP}{dr} = -Gm \quad \Big| \frac{d}{dr}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr} \quad \Big| \frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G \rho$$

use Polytrope — $P = K \cdot \rho^{\gamma}$, $\gamma = 1 + \frac{1}{n}$

Q: What are typical values of n?

Example: Fully convective stars (What... ... S = const

- of ideal gas: $n = 3/2$
- of rel. gas $n = 3$

$$\frac{(n+1)K}{4\pi G n} \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}} \cdot \frac{d\rho}{dr}\right) = -\rho$$

the function $\rho(r)$ is called the <u>Polytrope</u>

$$0 \leq r \leq R$$

<u>Boundary Conditions</u>

- surface : $R = r \longrightarrow \rho = 0$ because $P(R) = 0$
- centre : $\frac{dP}{dr} = 0 \longrightarrow \frac{d\rho}{dr} = 0 \quad |_{r=0}$

$\rightarrow$ uniquely define parameters K, n, R

<u>Ansatz</u> : <u>DEF</u> : $\Theta \ni \quad 0 \leq \Theta \leq 1$

with $\rho = \rho_c \cdot \Theta^n$

$$\rightarrow \underbrace{\left[\frac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}}\right]}_{=:\alpha^2} \cdot \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Theta}{dr}\right) = -\Theta^n$$

<u>DEF</u> : $r = \alpha \cdot \xi$

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\Theta}{d\xi}\right) = -\Theta^n}$$

Lane-Emden Equation!

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Boundary Conditions, in new variables

Surface: $\Theta \overset{!}{=} 0$

Centre: $\frac{d\Theta}{d\xi} = 0 \ (\xi = 0)$, $\Theta = 1$

Value of ξ at surface? need to find!

Def ξ_1 such that $\Theta(\xi_1) = 0$

Radius of Star:

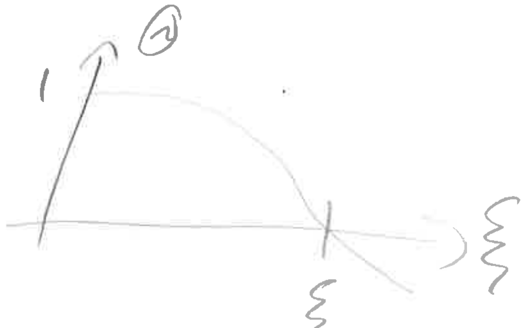
$[dr = \alpha \cdot d\xi]$

$$M = \int_0^R 4\pi r^2 \rho \, dr = 4\pi \alpha^3 \rho_c \int_0^{\xi_1} \xi^2 \Theta^n d\xi$$

USE LEQ:

$$M = -4\pi \alpha^3 \rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\Theta}{d\xi}\right) d\xi$$

$$\boxed{M = -4\pi \alpha^3 \rho_c \, \xi_1^2 \frac{d\Theta}{d\xi}\Big|_{\xi_1}}$$



Polytropic Constants:

$$R_n = \xi_1, \quad M_n = -\xi_1^2 \frac{d\Theta}{d\xi}\Big|_{\xi_1} > 0$$

CONST FOR SOLUTION TO LEQ FOR GIVEN n!

$\rightarrow R = R_n \cdot \alpha$, $M = \rho_c \cdot 4\pi \alpha^3 M_n$

Leaves α and ρ_c to depend on physical model

Mass-Radius-Relation

$$\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n} = \frac{[(n+1)K]^n}{4\pi G}$$

Special Case

$n=3 \rightarrow M$ independent of R!

$$M = 4\pi M_3 \left(\frac{K}{\pi G}\right)^{3/2}$$

Example: WD mass accretion

denser $\rightarrow$ more degenerate

EOS: non-rel : $n = 1.5$

rel : $n = 3$

more general Chandrasekhar EQN for WD

KW § 37

[typo in 37.13]

Recall

$$P_{e,\text{rel-deg}} = \underbrace{\frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\frac{1}{(\mu_e m_u)^{4/3}}}_{K_2} \cdot \rho^{4/3}$$

$$\rightarrow M = M_{CH} = \frac{M_3}{4\pi}\sqrt{\frac{3}{2}}\left(\frac{hc}{G m_u^{4/3}}\right)^{3/2} \cdot \mu_e^{-2} \simeq 5.83\,\mu_e^{-2} M_\odot$$

Q: μ_e? $\sim 0.5 \rightarrow M_{CH} \simeq 1.46 M_\odot$

Q: M_{CH} for "Fe" core?

More General:

$$M_{CRIT} \simeq M_{CH} \left[1 + \frac{\pi^2 k_B^2 T^2}{E_F^2} \right]$$

with $E_F = 1.11 \, \mathrm{MeV} \left[\frac{\rho}{10^7 \, g \, cm^{-3} \, mol} Y_e \right]^{1/3}$