Homework Set 2

Due: May 2, 2013, *before class*

1. **Stellar Collapse.**

   Assume a star initially in hydrostatic equilibrium collapses to a black hole. For simplicity, let's assume each shell collapses to the center in the free-fall time scale (dynamical time scale) given by

$$\tau_{\mathrm{ff}} = 1/\sqrt{24\pi G\bar{\rho}\,(m)}$$

   where $\bar{\rho}\,(m)$ is the average density inside mass coordinate m, and we neglect general relativity and pressure *during the collapse* (but not for the initial configuration of the star; "dust collapse").

   Compute the mass accretion rate onto the central black hole as a function of mass coordinate, m, density $\rho(m)$, and average enclosed density, $\bar{\rho}(m)$.

   (At what accretion rate would the shell of the star at mass coordinate m accrete onto the central black hole?)

   Using

$$\bar{\rho}(m) = 3m/\left(4\pi r^3\right)$$

   and

$$\mathrm{d}m = 4\pi r^2 \rho\, \mathrm{d}r$$

   we can write

$$\dot{M} = \left(\frac{\mathrm{d}\tau_{\mathrm{ff}}}{\mathrm{d}m}\right)^{-1}$$

$$\dot{M} = \sqrt{18G}\left(\frac{\mathrm{d}}{\mathrm{d}m}\left(\frac{m}{r^3}\right)^{-1/2}\right)^{-1}$$

$$\dot{M} = \sqrt{18G}\left(-\frac{1}{2}\left(\frac{m}{r^3}\right)^{-3/2}\left(\frac{1}{r^3} - \frac{3m}{r^4}\frac{\mathrm{d}r}{\mathrm{d}m}\right)\right)^{-1}$$

$$\dot{M} = 6\left(\frac{2Gm^3}{r^3}\right)^{1/2}\left(\frac{3m}{4\pi r^3\rho} - 1\right)^{-1}$$

$$\dot{M} = 6\left(\frac{2Gm^3}{r^3}\right)^{1/2}\left(\frac{\bar{\rho}}{\rho} - 1\right)^{-1}$$

$$\dot{M} = 6\,m\left(\frac{8\pi G\bar{\rho}}{3}\right)^{1/2}\left(\frac{\rho}{\bar{\rho}-\rho}\right)$$

$$\dot{M} = \frac{2m}{\tau_{\mathrm{ff}}}\left(\frac{\rho}{\bar{\rho}-\rho}\right) \quad \text{or} \quad \dot{M} = \frac{2m}{\tau_{\mathrm{ff}}}\left(\frac{\bar{\rho}}{\rho} - 1\right)^{-1}$$

   Ref: Woosley & Heger, ApJ, 752, 32, 2012.

Background: Consider the collapse of a massive star. When the very center collapses into a black hole, the rest of the star loses pressure support from below and collapses roughly on the free-fall time scale. In order to determine whether the star becomes a gamma-ray burst (GRB), we need to know the accretion rate onto the newly formed central black hole. Only if it is high enough will a gamma-ray burst result. Typically, the accretion rate needs to exceed some $0.1 - 1.0\,\mathrm{M}_{\odot}/\mathrm{yr}$. Knowing the accretion rate for each shell and the time for accretion of the shell, which is the free-fall time scale, we can now plot the accretion rate onto the black hole as a function of time, and estimate whether a gamma-ray burst should result or not.

As another note, if you consider a star of constant density, the average density, and hence the free-fall time, would be identical for all shells, i.e., for the entire star. Hence, at first there would be no accretion, then all would accrete in one instance, then there would be no more accretion.