EOS §4.1 §8.1, 13, 15, 16

optional §14 (IONIZATION)

EOS

What is EOS? | COMPOSITION + INDEP. VARIABLES e.g. T, S usually give P, E

What do we have to consider?

- GAS (ideal, degenerate) | composition ↖ what gases?
- RADIATION

→ change of constituents
→ ionization
→ nuclear reactions
→ strong, weak

chemical / nuclear equilibrium can be incorporated into EOS

Q: how to compute total pressure?

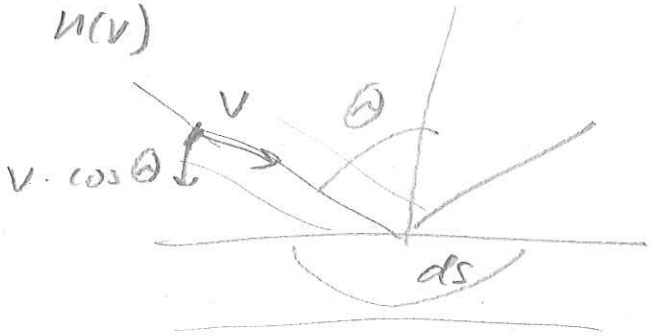
total pressure: $\sum P_i$

PARTIAL PRESSURE

Q: what path do we want to study in dependency?

$P = P_{RAD} + \underbrace{P_{IONS} + P_{electron}}_{P_{GAS}}$

PRESSURE



particles $\frac{2\pi \sin\theta \, d\theta}{4\pi} = \frac{1}{2}\sin\theta \, d\theta$ ← all possible directions

$$dp = 2\cos\theta$$

$$P = \underbrace{\int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta}_{1/3} \int_0^{\infty} n(p)\, v\, p\, dp = \frac{1}{3}\int_0^{\infty} n(p) p(p) dp$$

Energy density

$$u = \frac{1}{3}\int_0^{\infty} n(p)\, \varepsilon(p)\, dp$$

Q: Why do we integrate over p not over v?

Ions

$$P_I = n_I \cdot K T$$

Maxwell Boltzmann:

$$n(p)\,dp = \frac{n_I\, 4\pi p^2\, dp}{(2\pi m_I K T)^{3/2}}\, e^{-\frac{p^2}{2 m_I K T}}$$

What is n_I?

$$n_I = \sum n_i = \sum \frac{\rho}{u} \frac{X_i}{A_i} \quad \Big| \quad Y_i = \frac{X_i}{A_i}$$

X_i: mass fraction of i

A_i: mass / u

u: atomic mass number

DEF mean molecule weight of ions

$$\frac{1}{\mu_I} = \sum_i \frac{X_i}{A_i}$$

$$\rightarrow n_I = \frac{\rho}{\mu_I u} \quad ;$$

Gas constant $\mathcal{R} = \frac{K}{u} = K \cdot N_A$

$$P_I = \frac{\mathcal{R}}{\mu_I} \rho T$$

Electrons

$$P_e = n_e k T$$

$$n_e = \sum z \cdot n_i = \frac{\rho}{u} \sum x_i \frac{z_i}{A_i}$$

DEF mean molecular weight per electron

$$\frac{1}{\mu_e} = \sum_i x_i \frac{z_i}{A_i}$$

$$n_e = \frac{\rho}{\mu_e u}$$

Q Assume we have H: $A=1, z=1$

REST: $A = 2z$

→ DERIVE FORMULA FOR μ_e (or $\frac{1}{\mu_e}$)

$$P_e = \frac{\mathcal{R}}{\mu_e} \rho T$$

$$P_{GAS} = P_I + P_e = \mathcal{R} \rho T \underbrace{\left(\frac{1}{\mu_e} + \frac{1}{\mu_I} \right)}_{= \frac{1}{\mu}} = \frac{\mathcal{R} T \rho}{\mu}$$

$\Delta V \Delta^3 p \geq h^3$

DEG e^-

$$n_e(p)\,dp = \frac{2}{h^3} 4\pi p^2 dp \qquad p < p_0$$

$$p_f: \quad n_e = \int_0^{p_f} n_e(p)\,dp$$

$$\rightarrow p_f = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3}, \quad v = p/m_e$$

$$P_{e,deg} = \frac{8\pi}{15 m_e h^3} p_0^5 = \underbrace{\frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3} \frac{1}{u^{5/3}} \left(\frac{\rho}{\mu_e}\right)^{5/3}}$$

$$= K_1 \rho^{5/3}$$

REL. DEG: $v \rightarrow c$

$$P = \frac{1}{3}\int_0^\infty n(p)\, cp\, dp = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3} \frac{1}{u^{4/3}} \left(\frac{\rho}{\mu_e}\right)^{4/3}$$

$$=: K_2 \rho^{4/3}$$

Q when do we make transition to rel. deg regime ?

RADIATION

PLANCK: $u(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\frac{d\nu}{e^{\frac{h\nu}{kT}}-1}$

$$P = \frac{1}{3}\int_0^\infty c\,\frac{4U}{c}\,u(\nu)\,d\nu = \frac{1}{3}aT^4$$

$$a = \frac{8\pi^5 k^4}{15c^3h^3} = \frac{4\sigma}{c}$$

Energy Recall $u = \frac{1}{8}\int_0^\infty u(p)\,\varepsilon(p)\,dp$

REL E:

$$\varepsilon = m\cdot c^2\left[\left(1+\frac{p^2}{m^2c^2}\right)-1\right]$$

Q: limit $p \ll mc \longrightarrow p^2/2m$

non-rel

$$U_{GAS} = \frac{3}{2}\,\frac{P_{RAD}}{8}$$

rel: $U_{GAS} = 3\,\frac{P_{RAD}}{8}$

[RADIATION $\int_0^\infty u_\nu d\nu(\nu)\,d\nu = aT^4$]

$$U_{RAD} = \frac{aT^4}{8} = 3\,\frac{P_{RAD}}{8}]$$

→ NOTE: REL GAS & RAD GAS

Behave the same!

Q: what happens @ very high T?

Q : DERIVE Summary formula for Gas + electrons

$$\mu^{-1} = \sum_i \frac{z_i + 1}{A_i} x_i$$

Q Where are the different REGIMES IMPORTANT ?

ADIABATIC EXPANSION

$V = \frac{1}{\rho}$

$$du = T\,ds - P\,dV$$

adiabatic $\longleftrightarrow$ $dS = 0$

$$0 = du + P\,d\left(\frac{1}{\rho}\right)$$

[Q: What does adiabatic mean?]

in simple systems above $u \sim P/\rho$

$$\rightarrow u = \phi \cdot \frac{P}{\rho}$$

$$du = \phi \frac{1}{\rho} dP + \phi P\, d\left(\frac{1}{\rho}\right)$$

$$0 = \phi \frac{1}{\rho} dP + \phi P\, d\left(\frac{1}{\rho}\right) + P\, d\left(\frac{1}{\rho}\right)$$

$$\phi \frac{1}{\rho} dP + (1+\phi) P\, d\left(\frac{1}{\rho}\right) = 0$$

$$\phi \frac{1}{\rho} dP = (\phi+1) \frac{1}{\rho^2} P\, d\rho$$

$$\frac{\phi+1}{\phi} = \frac{\rho\, dP}{P\, d\rho} = \frac{d \ln P}{d \ln \rho}\bigg|_{ad} = \gamma_{ad}$$

Q compute adiabatic index for Ideal Gas, Real Gas, and Solid

other exponents

$$\frac{\Gamma_2}{\Gamma_2 - 1} : \left.\frac{d\ln P}{d\ln T}\right)_{ad} = \frac{1}{\nabla_{ad}}$$

$$\Gamma_3 = \left.\frac{d\ln T}{d\ln \rho}\right|_{ad}$$

Q: ARE ALL THREE INDEPENDENT? WHY?

$$\frac{\Gamma_1}{\Gamma_3 - 1} = \frac{\Gamma_2}{\Gamma_2 - 1}$$

CRYSTALLIZATION

COMPARE COULOMB ENERGY TO THERMAL E.

Thermal : KT

Coulomb $\frac{q^2}{d}$ $\qquad q = Z \cdot e$

d^3 : $\frac{4\pi r^3}{3}$: specific volume

VOLUME PER PARTICLE

$$V/m = \frac{V}{A \cdot u} = \frac{4\pi}{3} r^3 \simeq 4r^3$$

$$\Gamma = \frac{Z^2 e^2}{dKT}$$

Stellar structure

READ §1, 2.1–2.5, 2.7, 3, 4.2–4.4

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SUMMARY: §9 [NUM. SOL. §§10–12]

Stellar Stability §6 (BONUS §7)

Nuclear Reactions §8.2, 18.1(24) 5, 6

STRUCTURE / POLYTROPES §19 1–10