ADIABATIC EXPANSION

$V = \frac{1}{\rho}$

$$du = T\,ds - P\,dV$$

adiabatic $\longleftrightarrow$ $dS = 0$

$$0 = du + P\,d\left(\frac{1}{\rho}\right)$$

[Q: What does adiabatic mean?]

in simple systems above $u \sim P/\rho$

$$\rightarrow u = \phi \cdot \frac{P}{\rho}$$

$$du = \phi \frac{1}{\rho} dP + \phi P\, d\left(\frac{1}{\rho}\right)$$

$$0 = \phi \frac{1}{\rho} dP + \phi P\, d\left(\frac{1}{\rho}\right) + P\, d\left(\frac{1}{\rho}\right)$$

$$\phi \frac{1}{\rho} dP + (1+\phi) P\, d\left(\frac{1}{\rho}\right) = 0$$

$$\phi \frac{1}{\rho} dP = (\phi+1) \frac{1}{\rho^2} P\, d\rho$$

$$\frac{\phi+1}{\phi} = \frac{\rho\, dP}{P\, d\rho} = \frac{d \ln P}{d \ln \rho}\bigg)_{ad} = \gamma_{ad}$$

Q compute adiabatic index for Ideal Gas, Real Gas, and Solid

other exponents

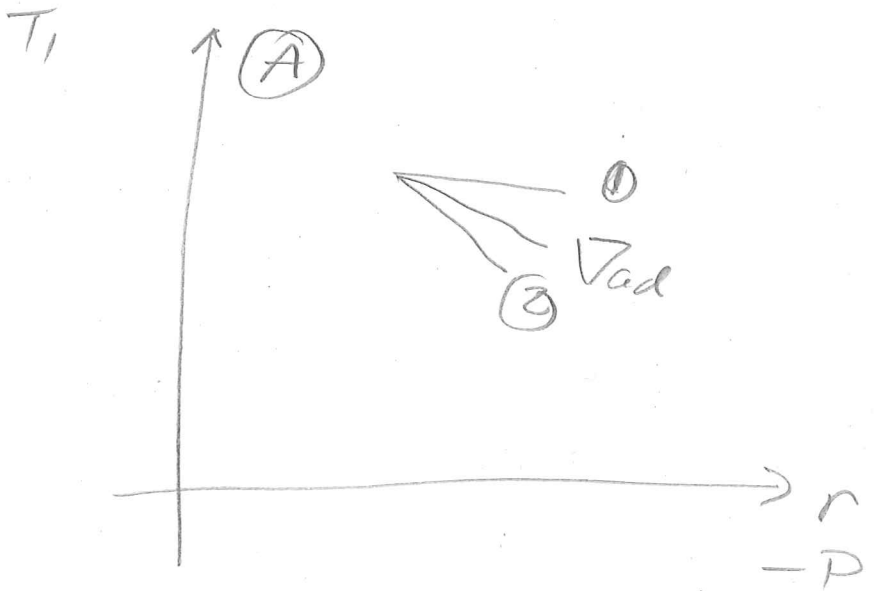
$$\frac{\Gamma_2}{\Gamma_2 - 1} : \left(\frac{d \ln P}{d \ln T}\right)_{ad} = \frac{1}{\nabla_{ad}}$$

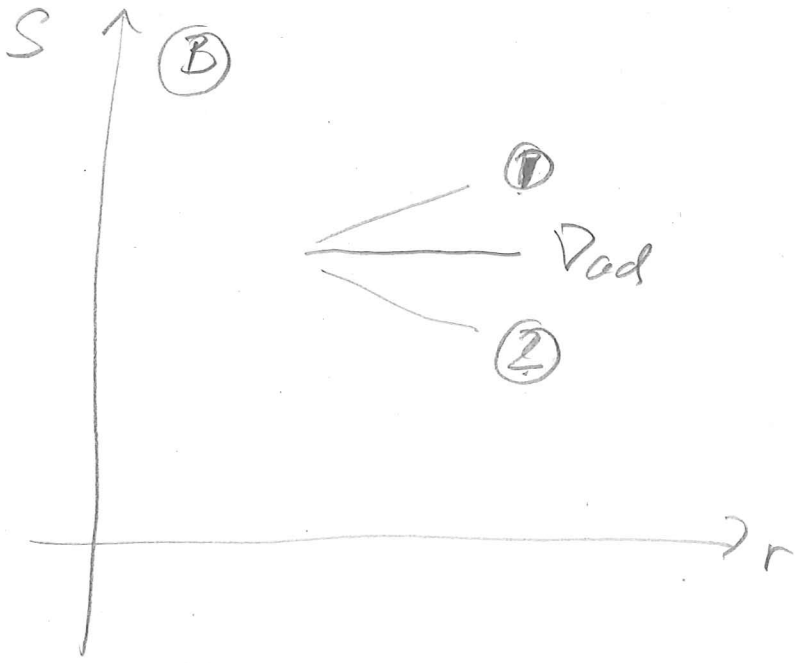
$$\Gamma_3 = \left(\frac{d \ln T}{d \ln \rho}\right)_{ad}$$

Q: ARE ALL THREE INDEPENDENT? WHY?

$$\frac{\Gamma_1}{\Gamma_3 - 1} = \frac{\Gamma_2}{\Gamma_2 - 1}$$

ADIABATIC GRADIENTS





WHICH ARE STABLE / UNSTABLE (convective)

Q: What are typical values of n?

Boundary conditions:

Surface: $\Theta = 0$

Core: $\frac{d\Theta}{d\xi} = 0 \quad (\xi = 0), \quad \Theta = 1$

Value of ξ_1 @ surface (need to find!)

ξ_1 such that $\Theta(\xi_1) = 0$

Radius of Star $R = \alpha \cdot \xi_1$

Mass of Star

$$M = \int_0^R 4\pi r^2 \rho \, dr = 4\pi \alpha^3 \rho_c \int_0^{\xi_1} \xi^2 \Theta^n \, d\xi$$

USE LEEQ

$$M = -4\pi \alpha^3 \rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\Theta}{d\xi}\right) d\xi = -4\pi \alpha^3 \rho_c \xi_1^2 \left.\frac{d\Theta}{d\xi}\right|_{\xi_1}$$

POLYTROPIC STELLAR MODELS (hydrostatic)

What is a polytrope? $P = K \cdot \rho^{\gamma}$

specific to model $\rightarrow P(r) = K \cdot \rho(r)^{\gamma}$

$\rightarrow T$-dep ~~implicit~~ (not incl.!) with const K, γ

different from Polytropic EOS $\boxed{P = K \cdot \rho^{\gamma}}$

START

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2} \quad \Big| \times \frac{r^2}{\rho}$$

$$\frac{r^2}{\rho}\frac{dP}{dr} = -Gm \quad \Big| \frac{d}{dr}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr} \quad \Big| \frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

use Polytropic EOS $P = K \cdot \rho^{\gamma}$

DEF $\gamma = 1 + \frac{1}{n}$

n: POLYTROPIC INDEX

$$\frac{(n+1)K}{4\pi G n}\,\frac{1}{r^2}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) = -\rho$$

$\rho(r)$ is called polytrope, $0 \le r \le R$

Boundary conditions

- Surface $R = r \rightarrow \rho = 0$ because $P(R) = 0$
- at the centre: $\frac{dP}{dr} = 0 \rightarrow \frac{d\rho}{dr} = 0 \;\Big|_{r=0}$

→ uniquely defined, Parameters K, n, R

Ansatz DEF $\Theta: \rightarrow 0 \le \Theta \le 1$

$$\rho = \rho_c \cdot \Theta^n$$

$$\rightarrow \underbrace{\left[\frac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}}\right]}_{=: \alpha^2} \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Theta}{dr}\right) = -\Theta^n$$

DEF $r = \alpha \cdot \xi$

$$\rightarrow \boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Theta}{d\xi}\right) = -\Theta^n}$$ LANE EMDEN EQUATION.

DEF POLYTROPIC CONSTANTS

$$R_n = -\xi_1$$

$$M_n = -\xi_1^2 \left(\frac{d\Theta}{d\xi}\right)_{\xi_1}$$

$$\rightarrow R = R_n \cdot \xi_1 \quad , \quad M = \rho_c 4\pi\alpha^2 M_n$$

→ DERIVE MASS–RADIUS RELATION

$$\left(\frac{GM}{M_n}\right)^{n-1} \left(\frac{R}{R_n}\right)^{3-n} = \frac{[(n+1)K]^n}{4\pi G}$$